Spring 2012

EE 330

ENGINEERING ELECTROMAGNETICS

HW 6: *Due Friday 17 February* (last HW before Exam #1*)*

2.58, 3.48, 3.58, 4.18, 4.23, 4.24, 4.31, 4.43, 4.48, 4.54, 4.56, 4.61

Problem 2.58 A lossless 100-Ω transmission line $3\lambda/8$ in length is terminated in an unknown impedance. If the input impedance is $Z_{\text{in}} = -j2.5\ \Omega$,

(a) Use the Smith chart to find Z_L.

(b) Verify your results using CD Module 2.6.

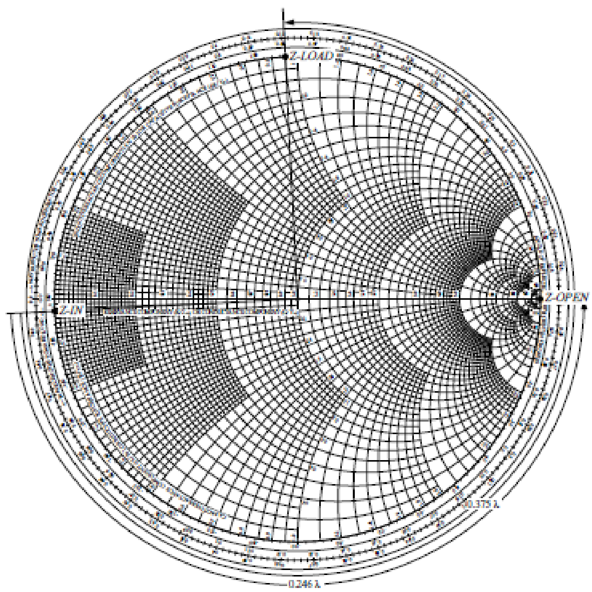


Figure P2.58: Solution of Problem 2.58.

Solution: Refer to Fig. P2.58. $z_{\text{in}} = Z_{\text{in}}/Z_0 = -j2.5\ \Omega/100\ \Omega = 0.0 - j0.025$ which is at point *Z-IN* and is at 0.004λ on the wavelengths to load scale.

(a) Point *Z-LOAD* is 0.375λ toward the load from the end of the line. Thus, on the wavelength to load scale, it is at $0.004\lambda + 0.375\lambda = 0.379\lambda$.

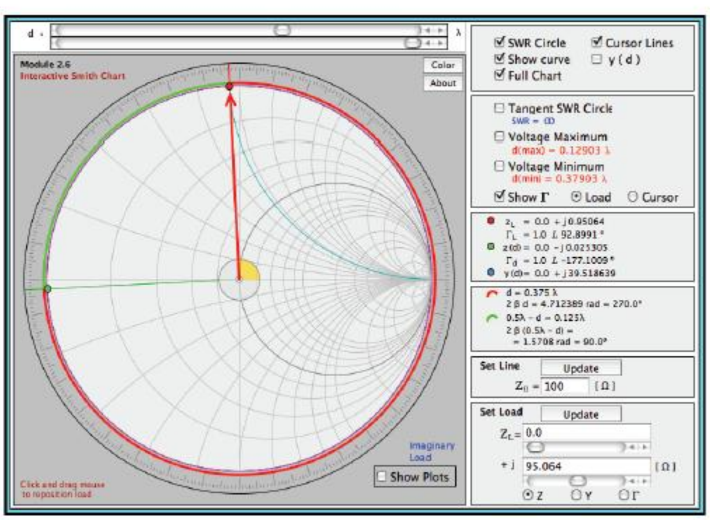
$$Z_L = z_L Z_0 = (0 + j0.95) \times 100\ \Omega = j95\ \Omega.$$

(b) After setting $d = 0.375\lambda$ in Module 2.6, the load point was moved over the circle to realize a value of $z(d) \simeq 0 - j0.025$. The corresponding value of z_L is:

$$z_L = 0 + j0.95064,$$

which gives

$$Z_L = (0 + j95)\ \Omega.$$



Problem 3.48 A vector field $\mathbf{D} = \hat{\mathbf{r}}r^3$ exists in the region between two concentric cylindrical surfaces defined by $r = 1$ and $r = 2$, with both cylinders extending between $z = 0$ and $z = 5$. Verify the divergence theorem by evaluating:

(a) $\oint_S \mathbf{D}\cdot d\mathbf{s}$,

(b) $\int_{\mathscr{V}} \nabla\cdot\mathbf{D}\,d\mathscr{V}$.

Solution:

(a)

$$\iint \mathbf{D}\cdot d\mathbf{s} = F_{\text{inner}} + F_{\text{outer}} + F_{\text{bottom}} + F_{\text{top}},$$

$$F_{\text{inner}} = \int_{\phi=0}^{2\pi}\int_{z=0}^{5} \left((\hat{\mathbf{r}}r^3)\cdot(-\hat{\mathbf{r}}r\,dz\,d\phi)\right)\Big|_{r=1}$$

$$= \int_{\phi=0}^{2\pi}\int_{z=0}^{5} \left(-r^4\,dz\,d\phi\right)\Big|_{r=1} = -10\pi,$$

$$F_{\text{outer}} = \int_{\phi=0}^{2\pi}\int_{z=0}^{5} \left((\hat{\mathbf{r}}r^3)\cdot(\hat{\mathbf{r}}r\,dz\,d\phi)\right)\Big|_{r=2}$$

$$= \int_{\phi=0}^{2\pi}\int_{z=0}^{5} \left(r^4\,dz\,d\phi\right)\Big|_{r=2} = 160\pi,$$

$$F_{\text{bottom}} = \int_{r=1}^{2}\int_{\phi=0}^{2\pi} \left((\hat{\mathbf{r}}r^3)\cdot(-\hat{\mathbf{z}}r\,d\phi\,dr)\right)\Big|_{z=0} = 0,$$

$$F_{\text{top}} = \int_{r=1}^{2}\int_{\phi=0}^{2\pi} \left((\hat{\mathbf{r}}r^3)\cdot(\hat{\mathbf{z}}r\,d\phi\,dr)\right)\Big|_{z=5} = 0.$$

Therefore, $\iint \mathbf{D}\cdot d\mathbf{s} = 150\pi$.

(b) From the back cover, $\nabla\cdot\mathbf{D} = (1/r)(\partial/\partial r)(rr^3) = 4r^2$. Therefore,

$$\iiint \nabla\cdot\mathbf{D}\,d\mathscr{V} = \int_{z=0}^{5}\int_{\phi=0}^{2\pi}\int_{r=1}^{2} 4r^2 r\,dr\,d\phi\,dz = \left(\left((r^4)\Big|_{r=1}^{2}\right)\Big|_{\phi=0}^{2\pi}\right)\Big|_{z=0}^{5} = 150\pi.$$

Problem 3.58 Find the Laplacian of the following scalar functions:

(a) $V_1 = 10r^3\sin 2\phi$

(b) $V_2 = (2/R^2)\cos\theta\sin\phi$

Solution:

(a)

$$\nabla^2 V_1 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_1}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V_1}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(10r^3\sin 2\phi)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}(10r^3\sin 2\phi) + 0$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(30r^3\sin 2\phi) - \frac{1}{r^2}(10r^3)4\sin 2\phi$$

$$= 90r\sin 2\phi - 40r\sin 2\phi = 50r\sin 2\phi.$$

(b)

$$\nabla^2 V_2 = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V_2}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V_2}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V_2}{\partial\phi^2}$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\left(\frac{2}{R^2}\cos\theta\sin\phi\right)\right)$$

$$+ \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(\frac{2}{R^2}\cos\theta\sin\phi\right)\right)$$

$$+ \frac{1}{R^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\left(\frac{2}{R^2}\cos\theta\sin\phi\right)$$

$$= \frac{4}{R^4}\cos\theta\sin\phi - \frac{4}{R^4}\cos\theta\sin\phi - \frac{2}{R^4}\frac{\cos\theta}{\sin^2\theta}\sin\phi$$

$$= -\frac{2}{R^4}\frac{\cos\theta\sin\phi}{\sin^2\theta}.$$

Problem 4.18 Multiple charges at different locations are said to be in equilibrium if the force acting on any one of them is identical in magnitude and direction to the force acting on any of the others. Suppose we have two negative charges, one located at the origin and carrying charge $-9e$, and the other located on the positive x-axis at a distance d from the first one and carrying charge $-36e$. Determine the location, polarity and magnitude of a third charge whose placement would bring the entire system into equilibrium.

Solution: If

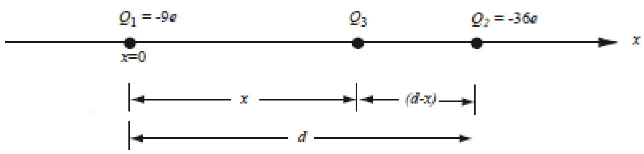


Figure P4.18: Three collinear charges.

$$\mathbf{F}_1 = \text{force on } Q_1,$$
$$\mathbf{F}_2 = \text{force on } Q_2,$$
$$\mathbf{F}_3 = \text{force on } Q_3,$$

then equilibrium means that

$$\mathbf{F}_1 = \mathbf{F}_2 = \mathbf{F}_3.$$

The two original charges are both negative, which mean they would repel each other. The third charge has to be positive and has to lie somewhere between them in order to counteract their repulsion force. The forces acting on charges Q_1, Q_2, and Q_3 are respectively

$$\mathbf{F}_1 = \frac{\hat{\mathbf{R}}_{21} Q_1 Q_2}{4\pi\varepsilon_0 R_{21}^2} + \frac{\hat{\mathbf{R}}_{31} Q_1 Q_3}{4\pi\varepsilon_0 R_{31}^2} = -\hat{\mathbf{x}}\frac{324e^2}{4\pi\varepsilon_0 d^2} + \hat{\mathbf{x}}\frac{9eQ_3}{4\pi\varepsilon_0 x^2},$$
$$\mathbf{F}_2 = \frac{\hat{\mathbf{R}}_{12} Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2} + \frac{\hat{\mathbf{R}}_{32} Q_3 Q_2}{4\pi\varepsilon_0 R_{32}^2} = \hat{\mathbf{x}}\frac{324e^2}{4\pi\varepsilon_0 d^2} - \hat{\mathbf{x}}\frac{36eQ_3}{4\pi\varepsilon_0 (d-x)^2},$$
$$\mathbf{F}_3 = \frac{\hat{\mathbf{R}}_{13} Q_1 Q_3}{4\pi\varepsilon_0 R_{13}^2} + \frac{\hat{\mathbf{R}}_{23} Q_2 Q_3}{4\pi\varepsilon_0 R_{23}^2} = -\hat{\mathbf{x}}\frac{9eQ_3}{4\pi\varepsilon_0 x^2} + \hat{\mathbf{x}}\frac{36eQ_3}{4\pi\varepsilon_0 (d-x)^2}.$$

Hence, equilibrium requires that

$$-\frac{324e}{d^2} + \frac{9Q_3}{x^2} = \frac{324e}{d^2} - \frac{36Q_3}{(d-x)^2} = -\frac{9Q_3}{x^2} + \frac{36Q_3}{(d-x)^2}.$$

Solution of the above equations yields

$$Q_3 = 4e, \qquad x = \frac{d}{3}.$$

Problem 4.23 Repeat Problem 4.22 for $\mathbf{D} = \hat{\mathbf{x}} xy^3z^3$ (C/m^2).

Solution:

(a) From Eq. (4.26), $\rho_v = \nabla \cdot \mathbf{D} = \frac{\partial}{\partial x}(xy^3z^3) = y^3z^3$.

(b) Total charge Q is given by Eq. (4.27):

$$Q = \int_{\mathcal{V}} \nabla \cdot \mathbf{D}\, d\mathcal{V} = \int_{z=0}^{2}\int_{y=0}^{2}\int_{x=0}^{2} y^3z^3\, dx\, dy\, dz = \left.\left.\left.\frac{xy^4z^4}{16}\right|_{x=0}^{2}\right|_{y=0}^{2}\right|_{z=0}^{2} = 32 \text{ C}.$$

(c) Using Gauss' law we have

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = F_{\text{front}} + F_{\text{back}} + F_{\text{right}} + F_{\text{left}} + F_{\text{top}} + F_{\text{bottom}}.$$

Note that $\mathbf{D} = \hat{\mathbf{x}} D_x$, so only F_{front} and F_{back} (integration over $\hat{\mathbf{z}}$ surfaces) will contribute to the integral.

$$F_{\text{front}} = \int_{z=0}^{2}\int_{y=0}^{2} \left.(\hat{\mathbf{x}} xy^3z^3)\right|_{x=2} \cdot (\hat{\mathbf{x}}\, dy\, dz)$$
$$= \int_{z=0}^{2}\int_{y=0}^{2} \left. xy^3z^3\right|_{x=2} dy\, dz = \left.\left(2\left(\frac{y^4z^4}{16}\right)\right|_{y=0}^{2}\right)\right|_{z=0}^{2} = 32,$$
$$F_{\text{back}} = \int_{z=0}^{2}\int_{y=0}^{2} \left.(\hat{\mathbf{x}} xy^3z^3)\right|_{x=0} \cdot (-\hat{\mathbf{x}}\, dy\, dz) = -\int_{z=0}^{2}\int_{y=0}^{2} \left. xy^3z^3\right|_{x=0} dy\, dz = 0.$$

Thus $Q = \oint \mathbf{D} \cdot d\mathbf{s} = 32 + 0 + 0 + 0 + 0 + 0 = 32$ C.

Problem 4.24 Charge Q_1 is uniformly distributed over a thin spherical shell of radius a, and charge Q_2 is uniformly distributed over a second spherical shell of radius b, with $b > a$. Apply Gauss's law to find $\mathbf{E}$ in the regions $R < a$, $a < R < b$, and $R > b$.

Solution: Using symmetry considerations, we know $\mathbf{D} = \hat{\mathbf{R}} D_R$. From Table 3.1, $d\mathbf{s} = \hat{\mathbf{R}} R^2 \sin\theta \, d\theta \, d\phi$ for an element of a spherical surface. Using Gauss's law in integral form (Eq. (4.29)),

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q_{\text{tot}},$$

where Q_{tot} is the total charge enclosed in S. For a spherical surface of radius R,

$$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} (\hat{\mathbf{R}} D_R) \cdot (\hat{\mathbf{R}} R^2 \sin\theta \, d\theta \, d\phi) = Q_{\text{tot}},$$

$$D_R R^2 (2\pi) [-\cos\theta]_0^\pi = Q_{\text{tot}},$$

$$D_R = \frac{Q_{\text{tot}}}{4\pi R^2}.$$

From Eq. (4.15), we know a linear, isotropic material has the constitutive relationship $\mathbf{D} = \varepsilon \mathbf{E}$. Thus, we find $\mathbf{E}$ from $\mathbf{D}$.

(a) In the region $R < a$,

$$Q_{\text{tot}} = 0, \qquad \mathbf{E} = \hat{\mathbf{R}} E_R = \frac{\hat{\mathbf{R}} Q_{\text{tot}}}{4\pi R^2 \varepsilon} = 0 \quad \text{(V/m)}.$$

(b) In the region $a < R < b$,

$$Q_{\text{tot}} = Q_1, \qquad \mathbf{E} = \hat{\mathbf{R}} E_R = \frac{\hat{\mathbf{R}} Q_1}{4\pi R^2 \varepsilon} \quad \text{(V/m)}.$$

(c) In the region $R > b$,

$$Q_{\text{tot}} = Q_1 + Q_2, \qquad \mathbf{E} = \hat{\mathbf{R}} E_R = \frac{\hat{\mathbf{R}} (Q_1 + Q_2)}{4\pi R^2 \varepsilon} \quad \text{(V/m)}.$$

Problem 4.31 The circular disk of radius a shown in Fig. 4-7 has uniform charge density ρ_s across its surface.

(a) Obtain an expression for the electric potential V at a point $P = (0,0,z)$ on the z-axis.

(b) Use your result to find $\mathbf{E}$ and then evaluate it for $z = h$. Compare your final expression with (4.24), which was obtained on the basis of Coulomb's law.

Solution:

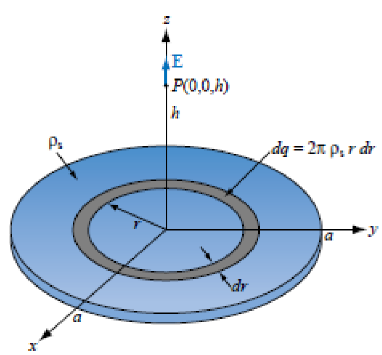


Figure P4.31: Circular disk of charge.

(a) Consider a ring of charge at a radial distance r. The charge contained in width dr is

$$dq = \rho_s (2\pi r \, dr) = 2\pi \rho_s r \, dr.$$

The potential at P is

$$dV = \frac{dq}{4\pi\varepsilon_0 R} = \frac{2\pi\rho_s r \, dr}{4\pi\varepsilon_0 (r^2 + z^2)^{1/2}}.$$

The potential due to the entire disk is

$$V = \int_0^a dV = \frac{\rho_s}{2\varepsilon_0} \int_0^a \frac{r \, dr}{(r^2+z^2)^{1/2}} = \frac{\rho_s}{2\varepsilon_0} (r^2 + z^2)^{1/2} \Big|_0^a = \frac{\rho_s}{2\varepsilon_0} \left[(a^2+z^2)^{1/2} - z \right].$$

(b)

$$\mathbf{E} = -\nabla V = -\hat{\mathbf{x}} \frac{\partial V}{\partial x} - \hat{\mathbf{y}} \frac{\partial V}{\partial y} - \hat{\mathbf{z}} \frac{\partial V}{\partial z} = \hat{\mathbf{z}} \frac{\rho_s}{2\varepsilon_0} \left[1 - \frac{z}{\sqrt{a^2+z^2}} \right].$$

The expression for $\mathbf{E}$ reduces to Eq. (4.24) when $z = h$.

Problem 4.43 A 100-m-long conductor of uniform cross-section has a voltage drop of 4 V between its ends. If the density of the current flowing through it is 1.4×10^6 (A/m^2), identify the material of the conductor.

Solution: We know that conductivity characterizes a material:

$$\mathbf{J} = \sigma \mathbf{E}, \qquad 1.4 \times 10^6 \text{ (A/m}^2) = \sigma \left(\frac{4 \text{ (V)}}{100 \text{ (m)}} \right), \qquad \sigma = 3.5 \times 10^7 \quad \text{(S/m)}.$$

From Table B-2, we find that aluminum has $\sigma = 3.5 \times 10^7$ (S/m).

Problem 4.48 With reference to Fig. 4-19, find $\mathbf{E}_1$ if $\mathbf{E}_2 = \hat{\mathbf{x}}3 - \hat{\mathbf{y}}2 + \hat{\mathbf{z}}2$ (V/m), $\varepsilon_1 = 2\varepsilon_0$, $\varepsilon_2 = 18\varepsilon_0$, and the boundary has a surface charge density $\rho_s = 3.54 \times 10^{-11}$ (C/m^2). What angle does $\mathbf{E}_2$ make with the z-axis?

Solution: We know that $\mathbf{E}_{1t} = \mathbf{E}_{2t}$ for any 2 media. Hence, $\mathbf{E}_{1t} = \mathbf{E}_{2t} = \hat{\mathbf{x}}3 - \hat{\mathbf{y}}2$. Also, $(\mathbf{D}_1 - \mathbf{D}_2) \cdot \hat{\mathbf{n}} = \rho_s$ (from Table 4.3). Hence, $\varepsilon_1(\mathbf{E}_1 \cdot \hat{\mathbf{n}}) - \varepsilon_2(\mathbf{E}_2 \cdot \hat{\mathbf{n}}) = \rho_s$, which gives

$$E_{1z} = \frac{\rho_s + \varepsilon_2 E_{2z}}{\varepsilon_1} = \frac{3.54 \times 10^{-11}}{2\varepsilon_0} + \frac{18(2)}{2} = \frac{3.54 \times 10^{-11}}{2 \times 8.85 \times 10^{-12}} + 18 = 20 \quad \text{(V/m)}.$$

Hence, $\mathbf{E}_1 = \hat{\mathbf{x}}3 - \hat{\mathbf{y}}2 + \hat{\mathbf{z}}20$ (V/m). Finding the angle $\mathbf{E}_2$ makes with the z-axis:

$$\mathbf{E}_2 \cdot \hat{\mathbf{z}} = |\mathbf{E}_2| \cos\theta, \qquad 2 = \sqrt{9+4+4} \cos\theta, \qquad \theta = \cos^{-1}\left(\frac{2}{\sqrt{17}}\right) = 61^\circ.$$

Problem 4.54 An electron with charge $Q_e = -1.6 \times 10^{-19}$ C and mass $m_e = 9.1 \times 10^{-31}$ kg is injected at a point adjacent to the negatively charged plate in the region between the plates of an air-filled parallel-plate capacitor with separation of 1 cm and rectangular plates each 10 cm^2 in area (Fig. P4.54). If the voltage across the capacitor is 10 V, find the following:

(a) The force acting on the electron.

(b) The acceleration of the electron.

(c) The time it takes the electron to reach the positively charged plate, assuming that it starts from rest.

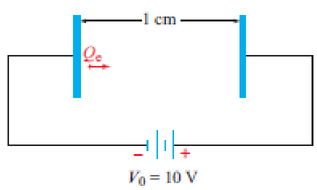


Figure P4.54: Electron between charged plates of Problem 4.54.

Solution:

(a) The electric force acting on a charge Q_e is given by Eq. (4.14) and the electric field in a capacitor is given by Eq. (4.112). Combining these two relations, we have

$$F = Q_e E = Q_e \frac{V}{d} = -1.6 \times 10^{-19} \frac{10}{0.01} = -1.6 \times 10^{-16} \quad \text{(N)}.$$

The force is directed from the negatively charged plate towards the positively charged plate.

(b)

$$a = \frac{F}{m} = \frac{1.6 \times 10^{-16}}{9.1 \times 10^{-31}} = 1.76 \times 10^{14} \quad \text{(m/s}^2).$$

(c) The electron does not get fast enough at the end of its short trip for relativity to manifest itself; classical mechanics is adequate to find the transit time. From classical mechanics, $d = d_0 + u_0 t + \frac{1}{2}at^2$, where in the present case the start position is $d_0 = 0$, the total distance traveled is $d = 1$ cm, the initial velocity $u_0 = 0$, and the acceleration is given by part (b). Solving for the time t,

$$t = \sqrt{\frac{2d}{a}} = \sqrt{\frac{2 \times 0.01}{1.76 \times 10^{14}}} = 10.7 \times 10^{-9} \text{ s} = 10.7 \quad \text{(ns)}.$$

Problem 4.56 Figure P4.56(a) depicts a capacitor consisting of two parallel, conducting plates separated by a distance d. The space between the plates contains two adjacent dielectrics, one with permittivity ε_1 and surface area A_1 and another with ε_2 and A_2. The objective of this problem is to show that the capacitance C of the configuration shown in Fig. P4.56(a) is equivalent to two capacitances in parallel, as illustrated in Fig. P4.56(b), with

$$C = C_1 + C_2 \tag{19}$$

where

$$C_1 = \frac{\varepsilon_1 A_1}{d} \tag{20}$$

$$C_2 = \frac{\varepsilon_2 A_2}{d} \tag{21}$$

To this end, proceed as follows:

(a) Find the electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$ in the two dielectric layers.

(b) Calculate the energy stored in each section and use the result to calculate C_1 and C_2.

(c) Use the total energy stored in the capacitor to obtain an expression for C. Show that (19) is indeed a valid result.

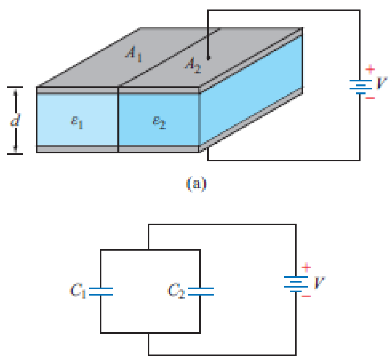
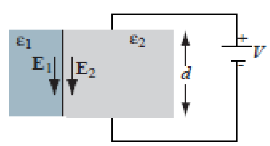


Figure P4.56: (a) Capacitor with parallel dielectric section, and (b) equivalent circuit.

Solution:



(c)

Figure P4.56: (c) Electric field inside of capacitor.

(a) Within each dielectric section, $\mathbf{E}$ will point from the plate with positive voltage to the plate with negative voltage, as shown in Fig. P4-56(c). From $V = Ed$,

$$E_1 = E_2 = \frac{V}{d}.$$

(b)

$$W_{e_1} = \frac{1}{2}\varepsilon_1 E_1^2 \cdot \mathcal{V} = \frac{1}{2}\varepsilon_1 \frac{V^2}{d^2} \cdot A_1 d = \frac{1}{2}\varepsilon_1 V^2 \frac{A_1}{d}.$$

But, from Eq. (4.121),

$$W_{e_1} = \frac{1}{2} C_1 V^2.$$

Hence $C_1 = \varepsilon_1 \frac{A_1}{d}$. Similarly, $C_2 = \varepsilon_2 \frac{A_2}{d}$.

(c) Total energy is

$$W_e = W_{e_1} + W_{e_2} = \frac{1}{2}\frac{V^2}{d}(\varepsilon_1 A_1 + \varepsilon_2 A_2) = \frac{1}{2} C V^2.$$

Hence,

$$C = \frac{\varepsilon_1 A_1}{d} + \frac{\varepsilon_2 A_2}{d} = C_1 + C_2.$$

Problem 4.61 With reference to Fig. P4.61, charge Q is located at a distance d above a grounded half-plane located in the x–y plane and at a distance d from another grounded half-plane in the x–z plane. Use the image method to

(a) Establish the magnitudes, polarities, and locations of the images of charge Q with respect to each of the two ground planes (as if each is infinite in extent).

(b) Find the electric potential and electric field at an arbitrary point $P = (0, y, z)$.

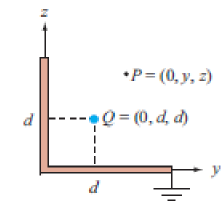


Figure P4.61: Charge Q next to two perpendicular, grounded, conducting half-planes.

Solution:

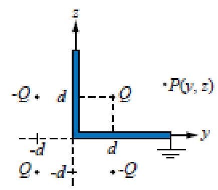


Figure P4.61: (a) Image charges.

(a) The original charge has magnitude and polarity $+Q$ at location $(0,d,d)$. Since the negative y-axis is shielded from the region of interest, there might as well be a conducting half-plane extending in the $-y$ direction as well as the $+y$ direction. This ground plane gives rise to an image charge of magnitude and polarity $-Q$ at location $(0,d,-d)$. In addition, since charges exist on the conducting half plane in the $+z$ direction, an image of this conducting half plane also appears in the $-z$ direction.

This ground plane in the x-z plane gives rise to the image charges of $-Q$ at $(0,-d,d)$ and $+Q$ at $(0,-d,-d)$.

(b) Using Eq. (4.47) with $N = 4$,

$$\begin{aligned}
V(x,y,z) &= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{|\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y-d)+\hat{\mathbf{z}}(z-d)|}-\frac{1}{|\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y+d)+\hat{\mathbf{z}}(z-d)|}\right.\\
&\qquad\left.+\frac{1}{|\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y+d)+\hat{\mathbf{z}}(z+d)|}-\frac{1}{|\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y-d)+\hat{\mathbf{z}}(z+d)|}\right)\\
&= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{\sqrt{x^2+(y-d)^2+(z-d)^2}}-\frac{1}{\sqrt{x^2+(y+d)^2+(z-d)^2}}\right.\\
&\qquad\left.+\frac{1}{\sqrt{x^2+(y+d)^2+(z+d)^2}}-\frac{1}{\sqrt{x^2+(y-d)^2+(z+d)^2}}\right)\\
&= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{\sqrt{x^2+y^2-2yd+z^2-2zd+2d^2}}\right.\\
&\qquad-\frac{1}{\sqrt{x^2+y^2+2yd+z^2-2zd+2d^2}}\\
&\qquad+\frac{1}{\sqrt{x^2+y^2+2yd+z^2+2zd+2d^2}}\\
&\qquad\left.-\frac{1}{\sqrt{x^2+y^2-2yd+z^2+2zd+2d^2}}\right) \quad \text{(V)}.
\end{aligned}$$

From Eq. (4.51),

$$\begin{aligned}
\mathbf{E} &= -\nabla V\\
&= \frac{Q}{4\pi\varepsilon}\left(\nabla\frac{1}{\sqrt{x^2+(y-d)^2+(z-d)^2}}-\nabla\frac{1}{\sqrt{x^2+(y+d)^2+(z-d)^2}}\right.\\
&\qquad\left.+\nabla\frac{1}{\sqrt{x^2+(y+d)^2+(z+d)^2}}-\nabla\frac{1}{\sqrt{x^2+(y-d)^2+(z+d)^2}}\right)\\
&= \frac{Q}{4\pi\varepsilon}\left(\frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y-d)+\hat{\mathbf{z}}(z-d)}{(x^2+(y-d)^2+(z-d)^2)^{3/2}}-\frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y+d)+\hat{\mathbf{z}}(z-d)}{(x^2+(y+d)^2+(z-d)^2)^{3/2}}\right.\\
&\qquad\left.+\frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y+d)+\hat{\mathbf{z}}(z+d)}{(x^2+(y+d)^2+(z+d)^2)^{3/2}}-\frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y-d)+\hat{\mathbf{z}}(z+d)}{(x^2+(y-d)^2+(z+d)^2)^{3/2}}\right) \quad \text{(V/m)}.
\end{aligned}$$